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Moment Approximation of Life Table Survival Model Using a Force of Mortality follow a Birth and Death Diffusion Process with General External Effect*

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ABSTRACT

One of the important functions of the demographers is to provide information on the trend of the life-table survival function, which is important to plan for human activities. Today, demographers are interested in describing phenomena in theoretical models involving population structure by considering the stochastic analogs of classical differences and differential equations. In this paper, the life-table survival function is considered using a force of mortality follow a birth and death diffusion process with general external effect process. The moment approximation as well as the mean and the variance of such a process are derived. Also, the moment approximation for some external effect distributions of Beta and Exponential, as well as for the case of no external effects are obtained. These results are useful in studying the behavior of the process and in statistical inference problems.

KEYWORDS - Life-Table Survival Function, Force of Mortality Process, Birth-Death Diffusion Process, General External Effect, Moment Approximation, Mean and Variance.

1. INTRODUCTION AND BACKGROUND

The life table is a statistical device that helps to present in an elegant and convenient way the information contained in a sequence of age specific rates. Also, life tables consist of different functions. For example, one of the most important functions is the survival function, which represents number of the survivors up to age. Thus, Ramakumar (1986) in his book described and studied all functions and tried to introduce deterministic analysis of survival function. Morton (1978) proposed a regression model for life tables and derived certain hypothesis testing procedures. Chiang (1960 a, b) introduced a stochastic analysis approach for the life table and its applications; especially he studied the sample variance of observed expectation of life table and other biometric functions.

Chiang (1968), Namboodiri (1991), Mitra (1983), Turner and Hanley (2010), Biswas (1988), Al-Eideh (2001), etc. have studied some other models for life-table survival function and other functions from various points of view.

In particular, Biswas (1988) discussed some gaps in the research and pedagogy of mathematical demography and other related topics in survival analysis by focusing intensively on a wide range of traditional as well as new inputs using a modern stochastic process and renewal theory oriented approach.

Al-Eideh (2001) studied the moment approximations as well as the mean and the variance of the life-table survival diffusion process with general external effect at a constant rate for some external effect distributions of Beta, Uniform and Exponential, and the case of no external effects.

Numerous researchers have worked on studying various problems related directly and indirectly to different functions of life table from different points of view. For example, Cohen (2010) studied new relationships for life expectancy based on the definition that the hazard of mortality is usually presented as a function of age, but can be defined as a function of the fraction of survivors. Specifically, in a life-table population with a positive age-specific force of mortality at all ages, the expectation of life at age is the average of the reciprocal of the survival-specific force of mortality at ages after, weighted by life-table deaths at each age after.

Pollard (2002) developed a simple exact formula for determining cohort life expectancies under constant continuous uniform improvement in mortality using only a cross-sectional (period) Gompertz life table for the lives concerned and a simple approximation applicable to all life tables. The present values of annuities for such lives can be determined simply and accurately across the whole age span.

According to Horiuchi and Wilmoth (1998), it is now considered as an established fact that mortality at advanced ages has a tendency to deviate from the Gompertz law, so that the logistic model often is used to fit human mortality. The estimates of mortality force at extreme ages are difficult because of small numbers of survivors to these ages in most countries. Data for extremely long-lived individuals are scarce and subjected to age exaggeration. More estimates that are accurate are obtained using the method of extinct generations (cf. Vincent (1951)) since traditional demographic estimates of mortality based on period data encounter well-known denominator problem.

This paper focusses on studying the moment approximations as well as the mean and the variance of the life-table survival function using a force of mortality follow a birth and death diffusion process with general external effect at a constant rate for some external effect distributions of Beta and Exponential, and the case of no effects.

The objective of this research is to identify critical knowledge types required by demographers of life-tables functions through building a stochastic survival model that has never been examined before as far as I know. The results should be very useful and will benefit the demographers and others to study the behavior of the number of survivors through different applications.

2. THE SURVIVAL MODEL USING A FORCE OF MORTALITY FOLLOW A BIRTH AND DEATH DIFFUSION PROCESS WITH GENERAL EXTERNAL EFFECT

Let $l(x)$ be the life-table survival function at age x. In probabilistic terms, $l(x)$ is one minus the cumulative distribution function of length of life x. Assume $l(x)$ is a continuous, differentiable function of x, then the age-specific force of mortality $\mu(x)$ at age *x* is defined by

$$\mu(x) = -\frac{1}{l(x)}\frac{dl(x)}{dx} \tag{1}$$

Solving the differential equation in equation (1), we get

$$l(x) = l(0)\exp\left(-\int_0^x \mu(x)dx\right) \tag{2}$$

Now, assuming

$$F(x) = \int_0^x \mu(x)dx \tag{3}$$

Substituting equation (3) in equation (2), we get

$$l(x) = l(0)\exp(-F(x)) \tag{4}$$

Where, $l(x)$ and $\mu(x)$ are the life-table survival function and the force of mortality at age x, respectively. Note that $l(0)$ refers to the initial number of survivors at age 0.

Now, assume the force of mortality $\{\mu(x); x \geq 0\}$ follow a birth and death diffusion process in which the diffusion coefficient a and the drift coefficient b are both proportional to $\mu(x)$ at age x. The diffusion process is assumed to be interrupted by external effects occurring at a constant rate c and having magnitudes with distribution $H(\cdot)$. Then $\{\mu(x); x \geq 0\}$ is a Markov process with State Space $S = [0, \infty)$ and can be regarded as a solution of the stochastic differential equation.

$$d\mu(x) = b\mu(x)dx + a\mu(x)dW(x) - \mu(x^-)dZ(x) \tag{5}$$

Here $\{W(x)\}$ is a Wiener process with mean zero and variance $\sigma^2 x$. Also, $\{Z(x)\}$ is a compound Poisson process

$$Z(x) = \sum_{i=1}^{N(x)} Y_i \tag{6}$$

Here $\{N(x)\}$ is a Poisson process with mean rate c, where c is the external jump rate, and $Y_1, Y_2, \ldots$, are independent and identically distributed random variables with distribution function $H(\cdot)$, with mean $m = E(Y_1)$ and variance $v^2 = Var(Y_1)$. Note that the moments of $Z(x)$ can be determined from the random sums formulas, and are

$$E[Z(x)] = cmx \tag{7}$$

and

$$Var[Z(x)] = c(v^2 + m^2)x \tag{8}$$

(cf. Taylor and Karlin (1984), pp. 55, 201)

Now from equation (5) we get,

$$\frac{d\mu(x)}{\mu(x)} = bdx + adW(x) - dZ(x) \tag{9}$$

Thus, the solution of the stochastic differential equation in (9) is given by

$$\mu(x) = \mu(0)\exp\{bx + aW(x) - Z(x)\} \tag{10}$$

where $\mu(0)$ is the initial force of mortality at age zero.

Now, solving $F(x)$ in equation (3), using Taylor and Karlin (1984, pp. 177) and Al-Eideh and Al-Hussainan (2002) and after some algebraic manipulations, it is easily shown that

$$\begin{aligned} F(x) &= \int_0^x \mu(s)ds = \int_0^x \mu(0)\exp\{bs + aW(s) - Z(s)\}ds \\ &= \frac{2(1-b)}{2a + a^2 - b^2}\mu(0)\exp\{bx + aW(x) - Z(x)\} \end{aligned} \tag{11}$$

Therefore, the life-table survival model $l(x)$ at age *x* using a force of mortality $\mu(x)$ follow the birth and death diffusion process defined in equation (4) is then given by

$$l(x) = l(0)\exp\left\{-\frac{2(1-b)}{2a + a^2 - b^2}\mu(0)\exp\{bx + aW(x) - Z(x)\}\right\} \tag{12}$$

where $l(0)$ and $\mu(0)$ are the initial number of survivors and the initial force of mortality at age zero respectively.

3. MOMENT APPROXIMATION OF THE LIFE TABLE SURVIVAL MODEL L(X) USING THE BIRTH AND DEATH DIFFUSION FORCE OF MORTALITY WITH EXTERNAL JUMP PROCESSES

In this section, the moment approximation as well as the mean and the variance for the survival model $l(x)$ using the force of mortality process $\mu(x)$ with external jump process $H(\cdot)$.

Let $M_n(x) = E\left[l^n(x)\right]$ for all $n = 1,2,3,\ldots,$ be the $n-th$ moment of the survival function $l(x)$. Then

$$M_n(x) = E\left[l^n(x)\right] = E\left[l^n(0)\exp(-nF(x)\right] = l^n(0)E\left[\exp(-nF(x)\right] \quad (13)$$

Note that

$$E\left[\exp(-nF(x)\right] \approx 1 - nE\left[F(x)\right] + \frac{n^2}{2}E\left[F^2(x)\right] \quad (14)$$

Using the results of finding the moment approximation of a birth and death diffusion process with general rate jump process (cf. Al-Eideh (2001)), it is easily shown that

$$\begin{aligned} E\left[F^n(x)\right] &= \left(\frac{2(1-b)}{2a+a^2-b^2}\right)^n \mu^n(0)E\left[\exp\{nbx + anW(x) - nZ(x)\}\right] \\ &= \left(\frac{2(1-b)}{2a+a^2-b^2}\right)^n \mu^n(0)E(\exp(nbx))E\left[\exp(naW(x))\right]E\left[\exp(-nZ(x))\right] \end{aligned} \quad (15)$$

Note that

$$E\left[\exp(naW(x)\right] = \exp\left(\frac{n^2}{2}a^2\sigma^2 x\right) \quad (16)$$

In addition, because of equation (14), we get

$$E\left[\exp(-nZ(x)\right] \approx 1 - nE\left[Z(x)\right] + \frac{n^2}{2}E\left[Z^2(x)\right] \quad (17)$$

Using equations (7) and (8), equation (17) becomes

$$E\left[\exp(-nZ(x)\right] \approx 1 - ncmx + \frac{n^2}{2}c\left(v^2 + m^2\right)x \quad (18)$$

Now, by direct substitution of equations (16) and (18), we obtain

$$\begin{aligned} E\left[F^n(x)\right] \approx{}& \mu^n(0)\left(\frac{2(1-b)}{2a+a^2-b^2}\right)^n \exp\left\{\left(nb + \frac{n^2}{2}a^2\sigma^2\right)x\right\} \\ &\cdot\left\{1 - ncmx + \frac{n^2}{2}c(v^2+m^2)x\right\} \end{aligned} \quad (19)$$

Consequently,

$$\begin{aligned} E\left[F(x)\right] \approx{}& \mu(0)\left(\frac{2(1-b)}{2a+a^2-b^2}\right)\exp\left\{\left(b + \frac{1}{2}a^2\sigma^2\right)x\right\} \\ &\cdot\left\{1 - cmx + \frac{1}{2}c(v^2+m^2)x\right\} \end{aligned} \quad (20)$$

And

$$\begin{aligned} E\left[F^2(x)\right] \approx{}& \mu^2(0)\left(\frac{2(1-b)}{2a+a^2-b^2}\right)^2 \exp\left\{\left(2b + 2a^2\sigma^2\right)x\right\} \\ &\cdot\left\{1 - 2cmx + 2c(v^2+m^2)x\right\} \end{aligned} \quad (21)$$

Therefore, by direct substitution of equations (20) and (21) in equation (14), we obtain from equation (13) the $n-th$ moment of the survival function $M_n(x) = E\left[l^n(x)\right]$ for all $n = 1,2,3,\ldots,$ is given by

$$M_n(x) = E\left[l^n(x)\right] \approx l^n(0)$$
$$- nl^n(0)\mu(0)\left(\frac{2(1-b)}{2a+a^2-b^2}\right)\exp\left\{\left(b+\frac{1}{2}a^2\sigma^2\right)x\right\}\left(1-cmx+\frac{1}{2}c\left(\nu^2+m^2\right)x\right) \tag{22}$$
$$+\frac{n^2}{2}l^n(0)\mu^2(0)\left(\frac{2(1-b)}{2a+a^2-b^2}\right)^2\exp\left\{\left(2b+2a^2\sigma^2\right)x\right\}\left(1-2cmx+2c\left(\nu^2+m^2\right)x\right)$$

where $l(0)$ and $\mu(0)$ are the initial number of survivors and the initial force of mortality at age zero respectively.

In particular, let $M_1(x) = E\left[l(x)\right]$ and $V(x) = V\left[l(x)\right]$ be the mean and the variance of $l(x)$ respectively. Then using equation (22), it is easily shown that

$$M_1(x) = E\left[l(x)\right] \approx l(0)$$
$$- l(0)\mu(0)\left(\frac{2(1-b)}{2a+a^2-b^2}\right)\exp\left\{\left(b+\frac{1}{2}a^2\sigma^2\right)x\right\}\left(1-cmx+\frac{1}{2}c\left(\nu^2+m^2\right)x\right) \tag{23}$$
$$+\frac{1}{2}l(0)\mu^2(0)\left(\frac{2(1-b)}{2a+a^2-b^2}\right)^2\exp\left\{\left(2b+2a^2\sigma^2\right)x\right\}\left(1-2cmx+2c\left(\nu^2+m^2\right)x\right)$$

And

$$M_2(x) = E\left[l^2(x)\right] \approx l^2(0)$$
$$- 2l^2(0)\mu(0)\left(\frac{2(1-b)}{2a+a^2-b^2}\right)\exp\left\{\left(b+\frac{1}{2}a^2\sigma^2\right)x\right\}\left(1-cmx+\frac{1}{2}c\left(\nu^2+m^2\right)x\right) \tag{24}$$
$$+2l^2(0)\mu^2(0)\left(\frac{2(1-b)}{2a+a^2-b^2}\right)^2\exp\left\{\left(2b+2a^2\sigma^2\right)x\right\}\left(1-2cmx+2c\left(\nu^2+m^2\right)x\right)$$

Therefore, the variance of $l(x)$ is then given by

$$V(x) = V\left[l(x)\right] = M_2(x) - \left(M_1(x)\right)^2 \tag{25}$$

where $M_1(x)$ and $M_2(x)$ are defined in equations (23) and (24) respectively.

4. EXAMPLES OF EXTERNAL EFFECT DISTRIBUTIONS

In this section, we will derive formulas for moment approximation for some external effect distributions.

4.1 Beta Distribution

Let $H(y)$ be the distribution function of beta random variable with parameters α and β, i.e.

$$H(y) = \begin{cases} 0, & y < 0 \\ \int_0^y \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}dx & 0 \le y < 1 \\ 1, & y \ge 1 \end{cases}$$

Where $B(\alpha,\beta)$ is the beta function, defined as

$$B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$$

Note that

$$m = \frac{\alpha}{\alpha+\beta} \qquad \text{and} \qquad \nu^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Now, substituting in equation (22), then the $n-th$ moment approximation of the survival function $M_n(x) = E[l^n(x)]$ for all $n = 1,2,3,\ldots,$ is given by

$$\begin{aligned} M_n(x) = E[l^n(x)] &\approx l^n(0) \\ &- nl^n(0)\mu(0)\left(\frac{2(1-b)}{2a+a^2-b^2}\right)\exp\left\{\left(b+\frac{1}{2}a^2\sigma^2\right)x\right\}\left(1-\frac{c\alpha}{\alpha+\beta}x+\frac{1}{2}\frac{c\alpha(\beta+1)}{\alpha+\beta+1}x\right) \\ &+ \frac{n^2}{2}l^n(0)\mu^2(0)\left(\frac{2(1-b)}{2a+a^2-b^2}\right)^2\exp\{(2b+2a^2\sigma^2)x\}\left(1-\frac{2c\alpha}{\alpha+\beta}x+\frac{2c\alpha(\beta+1)}{\alpha+\beta+1}x\right) \end{aligned} \quad (26)$$

where $l(0)$ and $\mu(0)$ are the initial number of survivors and the initial force of mortality at age zero respectively.

Note that when $\alpha = 1$ and $\beta = 1$, then $H(y)$ becomes

$$H(y) = \begin{cases} 0, & y < 0 \\ y & 0 \le y < 1 \\ 1, & y \ge 1 \end{cases}$$

Therefore, $H(y)$ is the distribution function of the uniform random variable on $[0,1]$. Thus, substituting in equation (26), we obtain the $n-th$ moment approximation of the survival function $M_n(x) = E[l^n(x)]$ for all $n = 1,2,3,\ldots,$ as follows

$$\begin{aligned} M_n(x) = E[l^n(x)] &\approx l^n(0) \\ &- nl^n(0)\mu(0)\left(\frac{2(1-b)}{2a+a^2-b^2}\right)\exp\left\{\left(b+\frac{1}{2}a^2\sigma^2\right)x\right\}\left(1-\frac{1}{6}cx\right) \\ &+ \frac{n^2}{2}l^n(0)\mu^2(0)\left(\frac{2(1-b)}{2a+a^2-b^2}\right)^2\exp\{(2b+2a^2\sigma^2)x\}\left(1+\frac{1}{3}cx\right) \end{aligned} \quad (27)$$

where $l(0)$ and $\mu(0)$ are defined above.

4.2 Exponential Distribution

Let $H(y)$ be the distribution function of exponential random variable with parameter λ, i.e.

$$H(y) = \begin{cases} 0, & y < 0 \\ 1-e^{-\lambda y}, & y \ge 0 \end{cases}$$

Note that

$$m = \frac{1}{\lambda} \qquad \text{and} \qquad \nu^2 = \frac{1}{\lambda^2}$$

Now, substituting in equation (22), then the $n-th$ moment of the survival function $M_n(x) = E[l^n(x)]$ for all $n = 1,2,3,\ldots,$ is given by

$$M_n(x) = E\left[l^n(x)\right] \approx l^n(0)$$
$$- nl^n(0)\mu(0)\left(\frac{2(1-b)}{2a+a^2-b^2}\right)\exp\left\{\left(b+\frac{1}{2}a^2\sigma^2\right)x\right\}\left(1-\frac{c}{\lambda}x+\frac{c}{\lambda^2}x\right) \quad (28)$$
$$+\frac{n^2}{2}l^n(0)\mu^2(0)\left(\frac{2(1-b)}{2a+a^2-b^2}\right)^2\exp\left\{\left(2b+2a^2\sigma^2\right)x\right\}\left(1-\frac{2c}{\lambda}x+\frac{4c}{\lambda^2}x\right)$$

where $l(0)$ and $\mu(0)$ are the initial number of survivors and the initial force of mortality at age zero respectively.

5. CASE OF NO EXTERNAL EFFECT

This case happens only when the external effect rate $c = 0$ and by substituting in equation (22), we obtain the $n-th$ moment approximation of the survival function $M_n(x) = E\left[l^n(x)\right]$ for all $n = 1,2,3,\ldots,$ as follows

$$M_n(x) = E\left[l^n(x)\right] \approx l^n(0)$$
$$- nl^n(0)\mu(0)\left(\frac{2(1-b)}{2a+a^2-b^2}\right)\exp\left\{\left(b+\frac{1}{2}a^2\sigma^2\right)x\right\} \quad (29)$$
$$+\frac{n^2}{2}l^n(0)\mu^2(0)\left(\frac{2(1-b)}{2a+a^2-b^2}\right)^2\exp\left\{\left(2b+2a^2\sigma^2\right)x\right\}$$

where $l(0)$ and $\mu(0)$ are the initial number of survivors and the initial force of mortality at age zero respectively.

6. CONCLUSION

In conclusion, this study provides a methodology for studying the behavior of the life-table survival function. More specifically, this study departs from the traditional before and after regression techniques and the time series analysis, and developed a survival function using a force of mortality model that explicitly accounts for the variations and volatilities in its values using a force of mortality follow a birth and death diffusion with general external effect processes. The moment approximations of such a process are derived for some external effect distributions of Beta and Exponential, and for the case of no external effects. In addition, some inference problems could be done for this model.

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Conflict of interests

The author declares that in the research and preparation of this article, there were no conflicts of interest related to certain organizations, institutions, individuals, or groups.

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